HOSSAM GHANEM

(18) 3.2 Definition of Derivative (B)

$$f^{\backslash}(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$f^{\backslash}(a) = \lim_{h\to 0}\frac{f(h+a)-f(a)}{h}$$

$$f^{\backslash}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example 1 44 November 9, 2006 15 July 1996	Use the definition of derivative to find $f^{\backslash}(x)$ where $f(x) = x + \sqrt{x}$

Solution

$$f^{\backslash}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{x+h+\sqrt{x+h}-x-\sqrt{x}}{h} = \lim_{h\to 0}\frac{h}{h} + \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= 1 + \lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = 1 + \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= 1 + \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = 1 + \lim_{h\to 0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)} = 1 + \frac{1}{2\sqrt{x}}$$

Example 2 August 7, 2011

3. (4 Points) Let $f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 3 & \text{if} \quad x \geq 0. \end{cases}$ Is f differentiable at $x = 0$? (justify your answer)

Solution

$\lim_{x\to 0^-} f(x) = 1$

$\lim_{x\to 0^+} f(x) = 3$

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

f discont. at $x = 0$

f not differentiable at $x = 0$

Example 3 24 November 3, 1998	Use the definition of derivative to find $f^{\backslash}(x)$ where $f(x) = x^2 + \sin x$

Solution

$f(x) = x^2 + \sin x$

$f(x+h) = (x+h)^2 + \sin(x+h)$

$$f^{\backslash}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + \sin(x+h) - x^2 - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} + \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} + \lim_{h\to 0} \frac{\sin x \cos h + \cos x \ \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} + \lim_{h\to 0} \frac{(\sin x \cos h - \sin x) + \cos x \ \sin h}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h} + \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \ \sin h}{h}$$

$$= \lim_{h\to 0}(2x+h) + \lim_{h\to 0} \sin x \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= 2x + \sin x\,(0) + \cos x\,(1) = 2x + 0 + \cos x = 2x + \cos x$$

DIFFERENTIABLE AND CONTINUOUS FUNCTIONS

Example 4	Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ A\,x + B & \text{if } x > 1 \end{cases}$ Find A and B so that $f^{\backslash}(x)$ is exist

Solution

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x^2 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} A\,x + B = A + B$

$\because$ f is differentiable $\therefore$ f is cont.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\therefore A + B = 1$

$B = 1 - A$

$$f^{\backslash}(1^-) = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} (x+1) = 2$$

$$f^{\backslash}(1^+) = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{A\,x + B - 1}{x - 1} = \lim_{x\to 1^+} \frac{A\,x + 1 - A - 1}{x - 1} = \lim_{x\to 1^+} \frac{A\,x - A}{x - 1}$$

$$= \lim_{x\to 1^+} \frac{A\,(x-1)}{x-1} = A$$

$\because$ f is differentiable $\therefore$ $f(1^-) = f(1^+)$

$\therefore A = 2$

$B = 1 - 2 = -1$

Example 5 1 December 3 1992	Let $f(x) = \begin{cases} x^2\sqrt{1+\dfrac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Use the definition to determine whether $f^\backslash(0)$exists or not

Solution

$$x^2\sqrt{1+\frac{1}{x^2}} = \frac{x^2}{|x|}\sqrt{x^2+1}$$

$$f(x) = \begin{cases} x\sqrt{x^2+1} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x\sqrt{x^2+1} & \text{if } x < 0 \end{cases}$$

$$f^\backslash(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$f^\backslash(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{-x\sqrt{x^2+1}}{x} = \lim_{x\to 0} -\sqrt{x^2+1} = -1$$

$$f^\backslash(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x\sqrt{x^2+1}}{x} = \lim_{x\to 0^+}\sqrt{x^2+1} = 1$$

$$\therefore\ f^\backslash(0^-) \neq f^\backslash(0^+) \quad \to \quad \therefore\ f^\backslash(0)\quad D.N.E$$

Example 6 13 November 13, 1995	Let $f(x) = \begin{cases} A\cos x + B\sin x & \text{if } x \geq 0 \\ A\,x^2 - x + 3B & \text{if } x < 0 \end{cases}$ Find the constants A and B for which $f^\backslash(0)$ exists

Solution

$f^\backslash(0)$ Exists

$\therefore\ f$ is cont $at\ x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} A\,x^2 - x + 3B = 3B$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} A\cos x + B\sin x = A$$

$$\therefore\ A = 3B$$

$$f(0) = A\cos(0) + B\sin(0) = A$$

$$f^\backslash(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{A\,x^2 - x + 3B - A}{x} = \lim_{x\to 0^-}\frac{A\,x^2 - x + A - A}{x} = \lim_{x\to 0^-}\frac{A\,x^2 - x}{x}$$

$$= \lim_{x\to 0^-}\frac{x(A\,x-1)}{x} = \lim_{x\to 0^-} Ax - 1 = -1$$

$$f^\backslash(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{A\cos x + B\sin x - A}{x} = \lim_{x\to 0^+}\frac{(A\cos x - A) + B\sin x}{x}$$

$$= \lim_{x\to 0^+}\frac{A(\cos x - 1)}{x} + \lim_{x\to 0^+}\frac{B\sin x}{x} = 0 + B = B$$

$f^\backslash(0)$ exists $\to f^\backslash(0^-) = f^\backslash(0^+)$

$B = -1$

$A = -3$

Example 7	Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0 \end{cases}$ Show that $f^{\backslash}(0)$ exists

Solution

$$f(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{0 - 0}{x} = 0$$

$$f(0^-) = \lim_{x \to 0^-} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0^-} x \sin \frac{1}{x}$$

$$-1 \leq \sin \tfrac{1}{x} \leq 1$$

$$x < 0$$

$$-x \geq x \sin \tfrac{1}{x} \geq x$$

$$\lim_{x \to 0^+} -x = \lim_{x \to 0^+} x = 0$$

$$f(0^{-1}) = \lim_{x \to 0^-} x \sin \frac{1}{x} = 0$$

$$\therefore \quad f(0^+) = f(0^-)$$

$$f^{\backslash}(0) = 0 \quad \rightarrow \quad \therefore f^{\backslash}(0) \text{ Exists}$$

Homework

1 12 November 1995	Use the definition of derivative to find $f^{\backslash}(x)$, where $f(x)=\sqrt{x+1}$ where $x>-1$
2	Let $f(x)=\sqrt{1+x^2}$. Find $f^{\backslash}(x)$ from the definition of derivative
3 42 March 29, 2006	Given $f(x)=\begin{cases} x^2-x+1 & \text{if} \quad x<1 \\ x^3 & \text{if} \quad 1\le x \end{cases}$ Find $f^{\backslash}(-1)$ and $f^{\backslash}(1)$ if they exists or explain why, if any of them does not exist
4	Let $f(x)=\begin{cases} \dfrac{x^2+1}{5} & \text{if} \quad x\ge 0 \\ \dfrac{8}{x^2+4} & \text{if} \quad x<0 \end{cases}$ Use the definition to find the derivative of f at $x=-2$ if it exists
5 6 January 6, 1993	Find the constants A and B so that $f(x)=\begin{cases} 2\sin x+\cos x & \text{if} \quad x<0 \\ A\,x+B & \text{if} \quad x\ge 0 \end{cases}$ is continuous and differentiable at $x=0$
6	Let $f(x)=\begin{cases} x^2+2x+1 & \text{if} \quad x\le 0 \\ 1+\sin Ax & \text{if} \quad x>0 \end{cases}$ Use the definition of derivative to find A so that f is differentiable at $x=0$
7	Test the continuity and differentiability of the function $f(x)=\begin{cases} x^2+2x+2 & \text{if} \quad x<0 \\ 2+\sin x & \text{if} \quad x\ge 0 \end{cases}$
8 11 August 11, 1994 A	Let $f(x)=\begin{cases} 3+\sin Ax & \text{if} \quad x\ge 0 \\ x^2+2x+3 & \text{if} \quad x<0 \end{cases}$ Use the definition of derivative to find A so that f is differentiable at $x=0$

Homework

9	Let $f(x)=\begin{cases}\sin\pi x, & x<1\\ A(x^2-1)+B, & x\ge 1.\end{cases}$ Find A and B so that $f^{\backslash}(1)$exists.
10 21 May 27. 2001	Let $f(x)=\begin{cases}\cos x & \text{, if } x<0,\\ 1-x^3 & \text{, if } x\ge 0.\end{cases}$ Use the definition of the derivative to find $f'(0)$.
11 19 July 29, 2000	Let $f(x)=\begin{cases}A+2x, & \text{if } x\le 1,\\ 2-Bx^2 & \text{if } x>1.\end{cases}$ Find all constants A and B so that f is differentiable at $x=1$
12 23 April 27, 2000	Let $f(x)=\begin{cases}\cos x, & x\ge 0\\ Ax+B, & x<0\end{cases}$ Find A and B so that f is differentiable at 0.
13 17 January 8 ,1997	Let h be differentiable on $(-\infty,\infty)$ and $h'(0)=2$ Find the constant A so that $f(x)=\begin{cases}\dfrac{h(x)-h(0)}{\sqrt{x+1}-1} & -1\le x<0\\ A & x\ge 0\end{cases}$ is continuous at $x=0$
14 25 January 12 .2003	Use the definition of the derivative to show that the following function is differentiable at $x=\frac{1}{2}$ $f(x)=\begin{cases}\cos(2x-1)+4x^2-2x & \text{, if } x\le \frac{1}{2}\\ 2-\dfrac{1}{2x} & \text{, if } x>\frac{1}{2}\end{cases}$

A 26 May 10, 2001	Use the definition of derivative to find $f^{\backslash}(1)$ if $f(x) = \sqrt[3]{x}$

Solution

$f(x) = \sqrt[3]{x}$

$f(1) = 1$

$$f^{\backslash}(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$f^{\backslash}(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x\to 1} \frac{x^{\frac{1}{3}} - 1}{x - 1} = \lim_{x\to 1} \frac{x^{\frac{1}{3}} - 1}{\left(x^{\frac{1}{3}} - 1\right)\left(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1\right)} = \lim_{x\to 1} \frac{1}{x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1} = \frac{1}{3}$$

B	Let $f(x) = \lvert x^2 - 1\rvert$. Discuss the differentiability of f at $x = 0$ and $x = 1$

Solution

$f(x) = \lvert x^2 - 1\rvert$

$$f^{\backslash}(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$f^{\backslash}(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{-(x^2 - 1) - 1}{x} = \lim_{x\to 0} \frac{-x^2 + 1 - 1}{x} = \lim_{x\to 0} -x = 0$$

$\therefore$ f is differentiable at $x = 0$

$$f^{\backslash}(1^-) = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x\to 1^-} \frac{-(x^2 - 1) - 0}{x - 1} = \lim_{x\to 1^-} \frac{-(x - 1)(x + 1)}{x - 1} = \lim_{x\to 1^-} -(x - 1) = -2$$

$$f^{\backslash}(1^+) = \lim_{x\to 1^+} \frac{-\lvert x^2 - 1\rvert - 0}{x - 1} = \lim_{x\to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^+} (x + 1) = 2$$

$f^{\backslash}(1^-) \neq f^{\backslash}(1^+)$

$\therefore$ f is not diferentiable at $x = 1$